Chapter 3

Properties of mixture copulas

3.1 Overview

The relationship between mixture copulas and their parents is explored in this chapter. First, an example is given to illustrate the process of parameter-mixing for copulas. Next, several ways in which mixing can potentially lead to modelling advantage are explored, including whether mixing leads to a functionally different copula; whether the parameter space can be extended; and whether mixing changes the dependence structures captured by the copula family. Also, the link between the mixture copulas and their parents is exploited to yield advantageous results useful in the application of mixture copulas.

3.2 Example of mixture copulas

The AMH family of 2-copulas (2.8) is indexed by values assigned to the dependence parameter $\theta \in [-1, 1]$, and is of the form

$$C_\theta(u, v) = \frac{uv}{1 - \theta(1 - u)(1 - v)}.$$

Consider, now, applying parameter mixing to this copula using a $Beta(\alpha, \beta)$ mixing distribution, which has pdf

$$f(x; \alpha, \beta) = \frac{x^{\alpha-1}(1 - x)^{\beta-1}}{B(\alpha, \beta)}, \qquad 0 < x < 1, \qquad \alpha > 0, \qquad \beta > 0,$$

where, for the avoidance of doubt, $B(\alpha, \beta)$ is used to represent the Beta function:

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1 - x)^{\beta-1} dx. \tag{3.1}$$

However, the domain of support of the $Beta$ distribution is $(0,1)$, while the parameter $\theta \in [-1,1]$. Thus, for the mixing distribution to cover the range of values of θ, a transformation is required. One such method is to let $X \sim \mathcal{F} = Beta(\alpha, \beta)$, so that $x \in (0,1)$. Then, for $\Theta = 2X-1$, $\theta \in (-1,1)$. Thus, the support of the transform $2X-1$ projects onto the same range of values as that of the dependence parameter θ. The distribution of the transform $2X-1$ can then be used as the mixing distribution to obtain the $Beta$-mixture family of AMH copula by applying (2.17) as follows:

$$
\begin{aligned}
C'_{\alpha,\beta}(u,v) &= C_\Theta(u,v) \underset{\Theta}{\wedge} (2X-1) \\
&= \int_0^1 \frac{uv}{1-(2x-1)(1-u)(1-v)} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} dx \\
&= \frac{1}{B(\alpha,\beta)} \frac{uv}{1+(1-u)(1-v)} \\
&\quad \times \int_0^1 \left(1 - \frac{2(1-u)(1-v)}{1+(1-u)(1-v)} x\right)^{-1} x^{\alpha-1}(1-x)^{(\alpha+\beta)-\alpha-1} dx \\
&= \frac{uv}{1+(1-u)(1-v)} \; {}_2F_1\left(1, \alpha; \alpha+\beta; s\right) \qquad (3.2) \\
&= \frac{uv}{1-(1-u)(1-v)} \; {}_2F_1\left(1, \beta; \alpha+\beta; \frac{s}{s-1}\right) \qquad (3.3)
\end{aligned}
$$

where

$$
s = \frac{2(1-u)(1-v)}{1+(1-u)(1-v)}
$$

is such that $0 \le s \le 1$ when $(u,v) \in \mathbb{I}^2$. The solution to the integral can be deduced as a special case of the single-valued, analytic definition of the Gaussian hypergeometric function given by Euler (e.g. see Rainville [1960, p.47]):

$$
{}_2F_1(p,q;r;s) = \frac{1}{B(q,r-q)} \int_0^1 (1-sx)^{-p} x^{q-1} (1-x)^{r-q-1} dx
$$

provided $|\arg(1-s)| < \pi$.

The last line of (3.3) is obtained by using Euler's transformation:

$$
{}_2F_1(p,q;r;s) = (1-s)^{-p} \; {}_2F_1\left(p, r-q; r; \frac{s}{s-1}\right).
$$

Note that

$$
\frac{s}{s-1} = \frac{2(1-u)(1-v)}{(1-u)(1-v)-1}.
$$

3.3 Equivalent functional form

In the interest of comparing modelling properties, it is readily obvious that nothing is gained by parameter-mixing if the mixture copula has the same form as the parent. This occurs in two types of cases.

The first case occurs when the parent copula $C_\theta(u,v)$ is linear in the parameter θ. In that case, the expectation in (2.17) applies directly to θ. Provided that $\mu_{\mathcal{F}} = E[\Theta]$ exists, the mixture copula will be equivalent in form to the parent copula, save that the parameter θ is replaced by its expectation, $\mu_{\mathcal{F}} = E[\Theta]$. In that case, no gain can be made from parameter-mixing. Indeed, if the parameter space induced through the new dependence parameter $\mu_{\mathcal{F}}$ exceeds the parameter space of θ, the additional parameters will not be identified.

To see this, consider the *Beta*-mixed FGM copula. The FGM family of 2-copulas (2.5) is indexed by a parameter $\theta \in [-1,1]$, and has the form

$$C_\theta(u,v) = uv + \theta uv(1-u)(1-v)$$

Then, applying (2.17), the *Beta*-mixture of the FGM copula is obtained as follows:

$$\begin{aligned} FGM \underset{\Theta}{\wedge} (2X-1) &= \int_0^1 uv(1+(2x-1)(1-u)(1-v))\frac{x^{\alpha-1}(1-x)^{\beta-1}}{Beta(\alpha,\beta)}dx \\ &= uv\left(1+\mu(1-u)(1-v)\right) \end{aligned}$$

where $\mu = \frac{\alpha-\beta}{\alpha+\beta}$. Clearly, the resultant mixture copula preserves the functional form, and thus the dependence structure, of the parent FGM copula. Although there are two parameters α and β, they cannot be separately identified. In fact, a re-parameterisation that sets $\alpha = \frac{1+\mu}{1-\mu}\beta$ would serve to eliminate the superfluous parameter β and isolate μ as the dependence parameter.

Secondly, even if the parent copula $C_\theta(u,v)$ is not linear in θ, there will be no gains made from parameter mixing if the mixture copula C'_λ is of the same family as C_θ. For example, consider Mardia's family of copulas,

$$C_\theta(u,v) = \frac{1}{2}\theta^2(1+\theta)M + (1-\theta^2)\Pi + \frac{1}{2}\theta^2(1-\theta)W \qquad -1 \le \theta \le 1$$

where M, W, and Π are the boundary and product copulas defined above. Being a linear and convex combination of the three non-parametric copulas (M, W, and Π), parameter mixing will result in another linear and convex combination of these three copulas. Despite being non-linear in θ, parameter-mixing applied to the Mardia family serves merely to recover the parent, differing only in its parameterisation.

Hence, if parameter-mixing is to result in any modelling advantage, it must first of all generate a family of copulas that is functionally different to the parent.

3.4 Identification

One way that the copula-based model can potentially be enhanced is by extending the parameter space, as in the *Beta-Binomial* example, so that the model enjoys enhanced flexibility. In order for parameter-mixing to successfully extend the parameter space, any additional induced parameters must be separately identified.

However, unlike the *Beta-Binomial* example, the parameter subject to mixing in the mixture copula is a dependence parameter. As a *convex* sum of the copulas in the parent family (see Section 2.7), the resultant copula must always retain the extremal forms as limiting cases. For copula families with continuous dependence coverage, this is a sign that dependence coverage remains the same. Hence, the additional parameters may not be identified.

As an example, consider the GB family of 2-copulas (2.9) which is indexed by values assigned to $\theta \in (0, 1]$ and has the form

$$C_\theta(u, v) = uv \exp\left(-\theta \ln u \ln v\right) \qquad \text{where } 0 < \theta \leq 1.$$

Clearly, $\lim_{\theta \to 0} C_\theta(u, v) = \Pi$ and $C_\theta(u, v) < \Pi$, so the GB family covers a region of negative dependence. To find the coverage in terms of Spearman's rho, we use (2.13):

$$\begin{aligned}
\rho_\theta &= 12 \int_{\mathbb{I}^2} C_\theta(u, v) du dv - 3 \\
&= 12 \int_0^1 \int_0^1 uv \exp\left(-\theta \ln u \ln v\right) du dv - 3 \\
&= 12 \int_0^1 v \left[\int_0^1 u^{1-\theta \ln v} du \right] dv - 3 \\
&= 12 \int_0^1 \frac{v}{2 - \theta \ln v} dv - 3.
\end{aligned}$$

Then, use the change of variable $v \to t$ such that $t = \frac{4}{\theta} - 2 \ln v$; then $v = \exp(-\frac{t}{2} + \frac{2}{\theta})$. The expression becomes:

$$\begin{aligned}
\rho_\theta &= 12 \int_{4/\theta}^{\infty} \frac{1}{\theta t} \exp(-\frac{t}{2} + \frac{2}{\theta}) \exp(-\frac{t}{2} + \frac{2}{\theta}) dt - 3 \\
&= 12\theta^{-1} e^{4/\theta} \int_{4/\theta}^{\infty} e^{-t} t^{-1} dt - 3 \\
&= 12\theta^{-1} e^{4/\theta} G\left(4/\theta\right) - 3 \qquad (3.4)
\end{aligned}$$

where $G(z) = \int_z^\infty e^{-t} t^{-1} dt$ is a special case of the incomplete gamma function such that, for $\theta \in (0, 1]$, $0 < G(4/\theta) \leq G(4) = 0.00378$.

The dependence coverage in terms of Spearman's rho is then $-0.523852 \leq \rho < 0$, where the lower bound is found by substitution of $\theta = 1$ into (3.4).

For a distribution $\mathcal{F}$ with pdf $f(\theta; \lambda)$, the $\mathcal{F}$-mix of the GB family of copulas is, applying (2.17),

$$\begin{aligned} C'_\lambda(u,v) &= C_\Theta(u,v) \underset{\Theta}{\wedge} \mathcal{F}(\lambda) \\ &= \Pi \int_0^1 \exp(-\theta(\ln u)(\ln v)) f(\theta;\lambda) d\theta \\ &= \Pi \ \mathrm{mgf}_{\mathcal{F}} \left(-(\ln u)(\ln v)\right) \end{aligned}$$

where $\mathrm{mgf}_{\mathcal{F}}$ denotes the moment generating function of $\mathcal{F}$, i.e. $E[\exp(t\Theta)]$.

Parameter-mixing in this instance produces a family different to the parent family. The question of interest is then whether the number of dependence parameters can be increased from the original singleton and still be formally identified. Let $\mathcal{F} = Beta(\alpha,\beta)$, with parameters $\alpha > 0$ and $\beta > 0$, and with mgf given by the confluent hypergeometric function ${}_1F_1\left(\alpha;\alpha+\beta;t\right)$, $t \in \mathbb{R}$; then the copula of the Beta parameter-mix of the GB family is given by,

$$\begin{aligned} C'_{\alpha,\beta}(u,v) &= \Pi \ {}_1F_1\left(\alpha;\alpha+\beta;-(\log u)(\log v)\right) \\ &= \Pi \exp(-(\log u)(\log v)) \ {}_1F_1\left(\beta;\alpha+\beta;(\log u)(\log v)\right) \end{aligned}$$

where the second line uses Kummer's relation

$${}_1F_1(p;q;x) = e^x \ {}_1F_1(q-p;q;-x).$$

For further details on the confluent hypergeometric function see, for example, Slater [1960].

In considering the identification of the parameters (α,β), it is important to note that limiting cases applied to the parameters correspond to the extremes of the dependence coverage of $C'_{\alpha,\beta}$. In this case, allowing α to be free and letting $\beta \to 0$ finds $C'_{\alpha,\beta}(u,v) \to C_1(u,v)$. Equally, $\alpha \to \infty$ and β free finds $C'_{\alpha,\beta}(u,v) \to C_1(u,v)$. Likewise, allowing α to be free but letting $\beta \to \infty$ finds $C'_{\alpha,\beta}(u,v) \to \Pi$, as too $\alpha \to 0$ and β free finds $C'_{\alpha,\beta}(u,v) \to \Pi$, as summarised below:

$$\begin{aligned} \lim_{\alpha\to 0} C'_{\alpha,\beta}(u,v) = \Pi, \qquad & \lim_{\beta\to 0} C'_{\alpha,\beta}(u,v) = C_1(u,v), \\ \lim_{\alpha\to\infty} C'_{\alpha,\beta}(u,v) = C_1(u,v), \qquad & \lim_{\beta\to\infty} C'_{\alpha,\beta}(u,v) = \Pi. \end{aligned}$$

That we cannot distinguish between the effect of, say, increasing α and decreasing β is an indication that these parameters will not be identified, and that the introduction of more parameters does not lead to added flexibility.

3.5 Dependence measure

Bearing in mind that parameter-mixing does not increase flexibility through extending the parameter space, the next issue is how mixing affects the dependence structures captured by the copula. This necessarily involves evaluating dependence measures for the mixture copula. In this respect, the link between the mixture copula and its parent copula can be exploited to computational advantage.

Because of the manner of their construction, certain properties of their parent copulas can be readily extended to the parameter-mixed copulas. One such property is the dependence measure Spearman's rho, which, for the copula $C_\theta(u,v)$, is

$$\rho = 12\int_{\mathbb{I}^2} C_\theta(u,v)dudv - 3 \tag{3.5}$$

Let $C'_\lambda(u,v)$ be a mixed copula as in (2.17). Let ρ_θ be the Spearman's rho measure for the parent family of copulas, and ρ'_λ be the Spearman's rho measure for the mixed copula . Then the Spearman's rho measure of the mixed copula can be derived by performing the parameter-mixing operation on the Spearman's rho measure of the parent copula. The proof is as follows. Starting with the expression for Spearman's Rho as above,

$$\begin{aligned}
\rho'_\lambda &= 12\int_{\mathbb{I}^2} C'_\lambda(u,v)dudv - 3 \\
&= 12\int_{\mathbb{I}^2}\left(\int_\Theta C_\theta(u,v)f(\theta;\lambda)d\theta\right)dudv - 3 \\
&= 12\int_\Theta\left(\int_{\mathbb{I}^2} C_\theta(u,v)dudv\right)f(\theta;\lambda)d\theta - 3 \\
&= 12\int_\Theta \frac{\rho_\theta + 3}{12} f(\theta;\lambda)d\theta - 3 \\
&= \int_\Theta \rho_\theta\, f(\theta;\lambda)d\theta \\
&= E_\theta\left[\rho_\theta\right]
\end{aligned} \tag{3.6}$$

where use has been made of Fubini's theorem. This result can be especially advantageous for copula families where Spearman's rho has a simple closed form, in which case (3.6) is a convenient method for deriving Spearman's rho expressions for the mixture copula — considerably easier than derivation from first principles.

To illustrate this, consider the Cuadras-Augé family of 2-copulas, which is indexed by parameter $\theta \in [0,1]$ and has the form:

$$C_\theta(u,v) = [\min(u,v)]^\theta[uv]^{1-\theta} = \begin{cases} uv^{1-\theta}, & u \le v, \\ u^{1-\theta}v, & u \ge v. \end{cases} \tag{3.7}$$

This family can be considered as a weighted geometric mean of M and Π, with $C_0 = \Pi$ and $C_1 = M$. The Cuadras-Augé family has a simple expression for Spearman's rho;

from (3.5):

$$\begin{aligned}
\rho_\theta &= 12\int_{\mathbb{I}^2} C_\theta(u,v)dudv - 3 \qquad (3.8)\\
&= 12\left(\int_{u\le v} uv^{1-\theta}dudv + \int_{u\ge v} u^{1-\theta}vdudv\right) - 3\\
&= 12\left(\int_0^1\int_0^v uv^{1-\theta}dudv + \int_0^1\int_0^u u^{1-\theta}vdvdu\right) - 3\\
&= \frac{3\theta}{4-\theta}
\end{aligned}$$

a relatively simple expression involving the parameter θ.

Consider now the Spearman's rho measure for the $Beta(\alpha,\beta)$-mixture of the Cuadras-Augé copula. Applying 3.6, this expression can simply be obtained as

$$\begin{aligned}
\rho'_{\alpha,\beta} &= E_\theta\left[\rho_\theta\right]\\
&= E_\theta[\frac{3\theta}{4-\theta}]\\
&= \int_0^1 \frac{3\theta}{4-\theta}\frac{\theta^{\alpha-1}(1-\theta)^{\beta-1}}{B(\alpha,\beta)}d\theta\\
&= \frac{3}{4}\frac{\alpha}{\alpha+\beta}\times\frac{1}{B(\alpha+1,\beta)}\int_0^1(1-\frac{1}{4}\theta)^{-1}\theta^\alpha(1-\theta)^{\beta-1}d\theta\\
&= \frac{3}{4}\frac{\alpha}{\alpha+\beta}\,{}_2F_1\left(1,\alpha+1;\alpha+\beta+1;\frac{1}{4}\right). \qquad (3.9)
\end{aligned}$$

The expression can be rewritten in various ways. First, use the following relation from Abramowitz and Stegun [1972, Equation (15.2.9)]:

$${}_2F_1\left(p,q;r;s\right) = \frac{-(r-q){}_2F_1(p,q-1;r;s)+(r-q){}_2F_1(p-1,q;r;s)}{(q-p)(1-s)},$$

and using ${}_2F_1\left(0,q;r;s\right)=1$, (3.9) can be written as:

$$\begin{aligned}
\rho'_{\alpha,\beta} &= \frac{3}{4}\frac{\alpha}{\alpha+\beta}\,{}_2F_1\left(1,\alpha+1;\alpha+\beta+1;\frac{1}{4}\right)\\
&= \frac{3}{4}\frac{\alpha}{\alpha+\beta}\frac{-4\beta\,{}_2F_1\left(1,\alpha;\alpha+\beta+1;\frac{1}{4}\right)+(\alpha+\beta)\,{}_2F_1\left(0,\alpha+1;\alpha+\beta+1;\frac{1}{4}\right)}{3\alpha}\\
&= 1-\frac{\beta\,{}_2F_1\left(1,\alpha;\alpha+\beta+1;\frac{1}{4}\right)}{\alpha+\beta}. \qquad (3.10)
\end{aligned}$$

Next, using the following relation from Abramowitz and Stegun [1972, Equation (15.2.21).]:

$${}_2F_1\left(p,q;r;s\right) = \frac{(r-1)(1-s){}_2F_1(p,q;r-1;s)+(r-p){}_2F_1(p-1,q;r;s)}{p-1-(r-q-1)s}$$

and ${}_2F_1\left(0,q;r;s\right)=1$, we can further write, from 3.10:

$$
\begin{aligned}
\rho'_{\alpha,\beta} &= 1-\frac{\beta\ {}_2F_1\left(1,\alpha;\alpha+\beta+1;\frac{1}{4}\right)}{\alpha+\beta} \\
&= 1-\beta\frac{\frac{3}{4}(\alpha+\beta){}_2F_1\left(1,\alpha;\alpha+\beta;\frac{1}{4}\right)-(\alpha+\beta){}_2F_1\left(0,\alpha;\alpha+\beta+1;\frac{1}{4}\right)}{-\frac{1}{4}\beta(\alpha+\beta)} \\
&= 1-\frac{4(\alpha+\beta)-3(\alpha+\beta)\ {}_2F_1\left(1,\alpha;\alpha+\beta;\frac{1}{4}\right)}{(\alpha+\beta)} \\
&= 3{}_2F_1\left(1,\alpha;\alpha+\beta;\frac{1}{4}\right)-3 \\
&= 4{}_2F_1\left[1,\beta;\alpha+\beta;-\frac{1}{3}\right]-3
\end{aligned}
$$

where the last line is obtained by using Euler's transformation:

$$
{}_2F_1\left(p,q;r;s\right)=(1-s)^{-p}\ {}_2F_1\left(p,r-q;r;\frac{s}{s-1}\right).
$$

As a side note, by substituting the limiting values of $\theta\in[0,1]$ into (3.8), we obtain the dependence coverage of the Cuadras-Augé in terms of Spearman's rho as $\rho\in[0,1]$. Likewise, by substituting the limiting values of α and β into 3.9, we find that:

$$
\begin{aligned}
\lim_{\alpha\to 0}\rho'_{\alpha,\beta}&=0, & \lim_{\beta\to 0}\rho'_{\alpha,\beta}&=1, \\
\lim_{\alpha\to\infty}\rho'_{\alpha,\beta}&=1, & \lim_{\beta\to\infty}\rho'_{\alpha,\beta}&=0,
\end{aligned}
$$

where use is made of the relations that

$$
{}_2F_1(p,q;r;s)=(1-s)^{-p} \qquad \text{for any } q=r
$$

and

$$
{}_2F_1\left(p,0;r;s\right)=1.
$$

These results illustrate two things. Firstly, the limits of the dependence coverage of the mixture copula, $\rho'_{\alpha,\beta}\in[0,1]$, are the same as the limits of the dependence coverage of the parent copula. This issue of dependence coverage is addressed in more detail in the next section. Secondly, increasing α while holding β constant traverses the dependence coverage, and so does decreasing β while holding α constant. This is an indication that α and β cannot be separately identified, just as in the Gumbel-Barnett example above.

3.6 Dependence structures covered

In the example of the AMH mixture given above (3.12), mixing yielded a copula with a different form to the parent. The question, then, is whether, and how, the dependence structures covered by the parent differ with those covered by the mixture copula.

To investigate this, first compare the dependence structures covered by the mixture copula as compared to the parent copula. In order to observe some of the effects of mixing, first consider the dependence coverage of the *Beta*-mixture family of AMH copulas. Now, the limiting forms of the parent, the AMH family of copulas, can be obtained by substituting the limits of the dependence parameter into (2.8) to obtain:

$$C_{+1}(u,v) = \frac{uv}{1-(1-u)(1-v)}$$

and

$$C_{-1}(u,v) = \frac{uv}{1+(1-u)(1-v)}$$

as the upper and lower limits respectively. Note that

$$C_{-1}(u,v) = \frac{\Pi}{2-\Sigma+\Pi}$$

and

$$C_{+1}(u,v) = \frac{\Pi}{\Sigma-\Pi}$$

where $\Sigma = u+v$.

Now consider the limiting forms of the *Beta*-mixture family of AMH copulas. First, consider the effect of varying α while keeping β fixed. Observe that, from (3.2),

$$C'_{\alpha}(u,v;\beta) = C_{-1}(u,v)\ {}_2F_1\left(1,\alpha;\alpha+\beta;s\right)$$

with, for $0<\alpha<\infty$, the Gaussian hypergeometric function satisfying

$$1 < {}_2F_1\left(1,\alpha;\alpha+\beta;s\right) < {}_1F_0\left(1;s\right) \tag{3.11}$$

where

$${}_1F_0\left(1;s\right) = (1-s)^{-1} = \frac{1+(1-u)(1-v)}{1-(1-u)(1-v)} = \frac{C_{+1}(u,v)}{C_{-1}(u,v)}. \tag{3.12}$$

For fixed u, v and β, ${}_2F_1\left(1,\alpha;\alpha+\beta;s\right)$ takes smaller (larger) values corresponding to smaller (larger) values of α. If the limit cases corresponding to $\alpha\to 0$ and $\alpha\to\infty$ are included, the range of dependence coverage of the mixed family of copulas is, as expected, equivalent to that of the parent family. To see this, substitute (3.12) into (3.11), and multiply through by $C_{-1}(u,v)$ to obtain:

$$C_{-1}(u,v) \le C'_{\alpha}(u,v;\beta) \le C_{+1}(u,v).$$

This result is illustrated in Figure 3.1 via a series of contour plots. As can be observed, the contour plot for the AMH copula with $\theta\to -1$ is identical to that for the *Beta*-mixture copula with $\alpha\to 0$. Likewise, the contour plot for the AMH copula with $\theta\to 1$ is similar to that for the *Beta*-mixed copula with α becoming large.

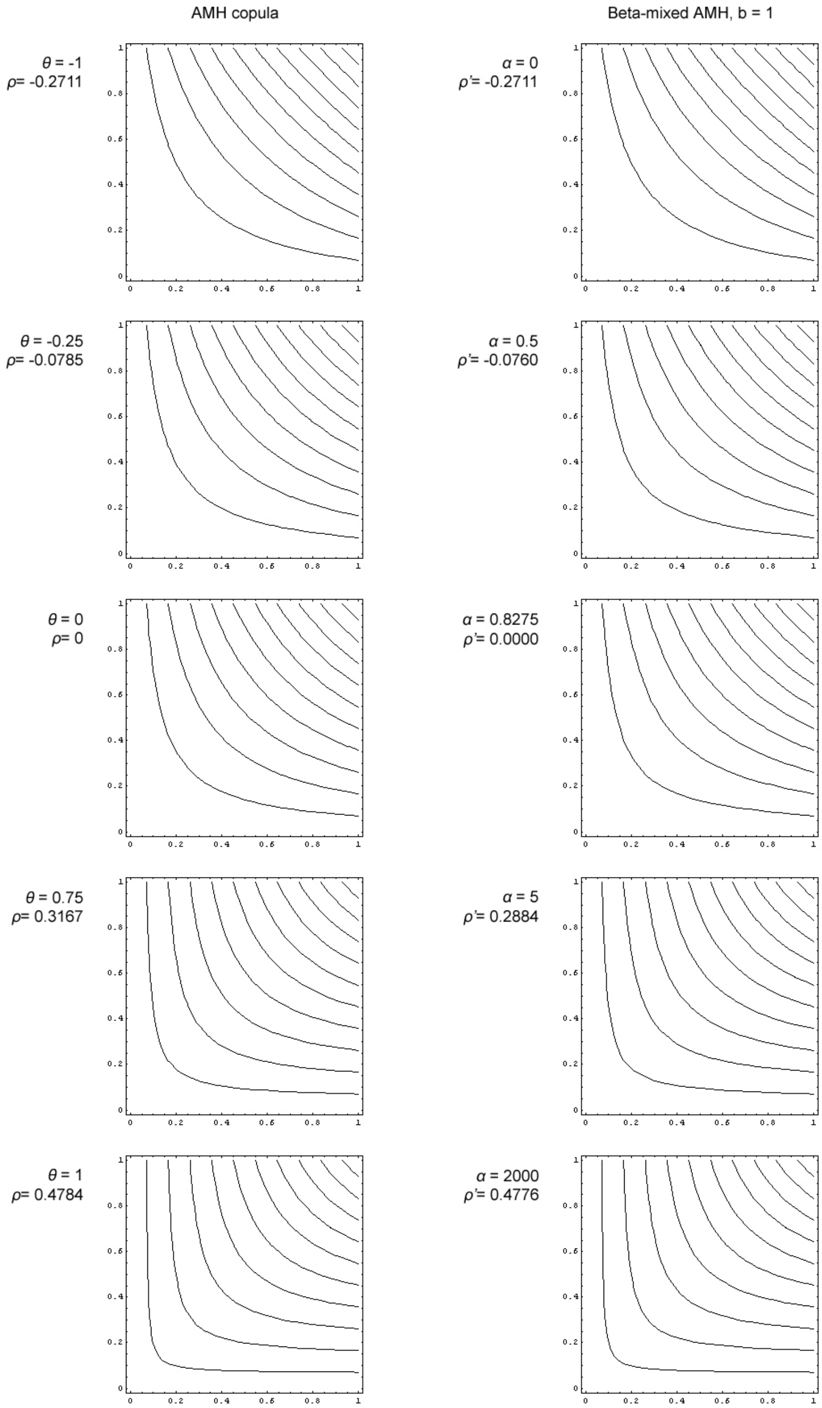

Figure 3.1: Contour plots of the AMH family and the Beta-mixed AMH family with known $b = 1$

This observation, that the mixture copula shares the same limiting forms as the parent copula can be confirmed via an examination of dependence statistics. As noted above at (2.15), Spearman's rho for the AMH family of copulas can be written as

$$\rho_\theta = \frac{12(1+\theta)}{\theta^2}\operatorname{di}\log(1-\theta) - \frac{24(1-\theta)}{\theta^2}\ln(1-\theta) - \frac{3(\theta+12)}{\theta}$$

and has values $\rho_\theta \in [-0.2711, 0.4784]$. To obtain Spearman's rho values for the *Beta*-mixture AMH copula, first observe that the AMH copula can be written as

$$C_\theta = uv\sum_{i=0}^{\infty}[\theta(1-u)(1-v)]^i \tag{3.13}$$

as a power series expansion of (2.8). Substituting this into (2.13), the formula for Spearman's rho, gives

$$\begin{aligned}
\rho_\theta &= 12\int_0^1\int_0^1 uv\sum_{i=0}^{\infty}[\theta(1-u)(1-v)]^i du dv - 3 \\
&= -3 + 12\sum_{i=0}^{\infty}\theta^i\int_0^1\int_0^1 uv[(1-u)(1-v)]^i du dv \\
&= -3 + 12\sum_{i=0}^{\infty}\theta^i\int_0^1 u(1-u)^i du\int_0^1 v(1-v)^i dv \\
&= -3 + 12\sum_{i=0}^{\infty}\theta^i\left[\int_0^1\left[(1-u)^{i+1}-(1-u)^i\right]du\right]^2 \\
&= -3 + 12\sum_{i=0}^{\infty}\theta^i\frac{1}{[(1+i)(2+i)]^2} \\
&= -3 + 12\sum_{i=0}^{\infty}\frac{\theta^i}{(2+3i+i^2)^2} \qquad (3.14)
\end{aligned}$$

which must be equivalent to (2.15). Then, applying the result relating to Spearman's rho for mixture copulas (3.6), we have

$$\begin{aligned}
\rho'_{\alpha,\beta} &= -3 + 12\sum_{i=0}^{\infty}\frac{(-1)^i\ {}_2F_1(\alpha,-i;\alpha+\beta;2)}{(2+3i+i^2)^2} \qquad (3.15) \\
&= -3 + 12\sum_{i=0}^{\infty}\frac{{}_2F_1(\beta,-i;\alpha+\beta;2)}{(2+3i+i^2)^2}. \qquad (3.16)
\end{aligned}$$

By substituting the limits of $\alpha \in (0,\infty)$, we find that $\rho'_{\alpha,\beta} \in (-0.2711, 0.4784)$. The same bounds are obtained by substituting the limits of $\beta \in (0,\infty)$. To all intents and purposes, this covers the same range of association as the parent AMH family of copulas. Also, that changing α while holding β constant produces the same dependence range as changing β while holding α constant is, again, indication that the two parameters are not, in fact, separately identified.

However, possessing the same range of association measures does not mean that the two families cover the same dependence structures. To see this, consider the AMH copula (2.8) with parameter $\theta = 0$, which is simply the Product copula:

$$C_0(u, v) = uv = \Pi.$$

with Spearman's rho $\rho_0 = 0$. However, while the AMH family nests the Product copula as a special case, the *Beta*-mixture family (3.3) does not. For $C'_{\alpha,\beta}(u, v)$ to nest Π, the Gaussian hypergeometric function ${}_2F_1(1, \alpha; \alpha + \beta; s)$ must be simplifiable to $(1-\frac{1}{2}s)^{-1} = 1+(1-u)(1-v)$ for some α at every given β, and vice versa. However, no such set of pairings can be found. This implies that even though the point at which the mixture copula transits from negative to positive dependence corresponds to zero dependence, that point does not represent independence; it is not Π.

Hence, the AMH family and the *Beta*-mixture family are non-nested with respect to each other. Despite having the same limiting forms, they describe different dependence structures.

3.7 Informative mixing

One additional advantage of mixture copulas is its ability to incorporate prior information. The hierarchical model under which parameter-mixing arises can be considered as a process of incorporating additional information concerning the parameter θ into the copula $C_\theta(u, v)$. Hence, the choice of mixing distribution can be a tool for conveying prior information, via a process here termed informative mixing.

In view of the results relating to the lack of identification in such mixtures detailed above, we will henceforth set $\beta = b$, a known constant, so that the $Beta(\alpha, b)$-mixture family of the AMH copula has only one parameter, $\alpha \in (0, \infty)$, and has the following functional form:

$$C'_\alpha(u, v; b) = \frac{uv}{1 + (1-u)(1-v)} \, {}_2F_1(1, \alpha; \alpha + b; s) \tag{3.17}$$

$$= \frac{uv}{1 - (1-u)(1-v)} \, {}_2F_1\left(1, b; \alpha + b; \frac{s}{s-1}\right) \tag{3.18}$$

where

$$s = \frac{2(1-u)(1-v)}{1 + (1-u)(1-v)}.$$

To illustrate informative mixing, consider the AMH family of copulas (2.8), which, as illustrated above, produces through mixing with a Beta distribution the $Beta(\alpha, \beta)$-mixture family of the AMH copula as in (3.2). To convey a prior expectation that the dependence structure is *positive*, we can construct a positive informative mixture; that is, a mixture that restricts attention to only the positive range of dependence. Instead of using the transform $2X-1$, where $X \sim Beta(\alpha, \beta)$ as in (3.3), mixture is simply taken with respect to X. This means that the support of the mixing distribution projects to only the positive portion of the support of the dependence parameter Θ. In other words, the mixing distribution has pdf

$$f(\theta) = \begin{cases} \frac{\theta^{\alpha-1}(1-\theta)^{b-1}}{Beta(\alpha,b)} & 0 \leq \theta \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

The positive informative *Beta*-mixture of the AMH copula (hereafter denoted the $Beta^+$-mixture) is, then:

$$\begin{aligned} C'^{+}_{\alpha}(u,v;b) &= C_\Theta(u,v) \underset{\Theta}{\wedge} X \\ &= \int_0^1 \frac{uv}{1-x(1-u)(1-v)} \frac{x^{\alpha-1}(1-x)^{b-1}}{Beta(\alpha,b)} dx \\ &= uv \; {}_2F_1\left(1, \alpha; \alpha+b; (1-u)(1-v)\right) \end{aligned} \tag{3.19}$$

The limiting forms of this distribution can be found by substituting limiting values of the parameter $\alpha > 0$. Using, from (3.11),

$$1 < \; {}_2F_1\left(1, \alpha; \alpha+b; s\right) < {}_1F_0\left(1; s\right)$$

we find that, $\lim_{\alpha\to 0} C'^{+}_{\alpha}(u,v;b) = \Pi$ and $\lim_{\alpha\to\infty} C'^{+}_{\alpha}(u,v;b) = C_{+1}(u,v)$, and so the $Beta^+$-mixture has dependence coverage as follows:

$$\Pi < C'^{+}_{\alpha}(u,v;b) < C_{+1}(u,v),$$

which captures only *positive* dependence.

Similarly, a negative-coverage mixture copula ($Beta^-$-mixture) can be constructed by reflecting the mixing Beta distribution about the origin, to produce:

$$C'^{-}_{\alpha}(u,v;b) = C_\Theta(u,v) \underset{\Theta}{\wedge} (-X) \tag{3.20}$$

$$\begin{aligned} &= \int_0^1 \frac{uv}{1+x(1-u)(1-v)} \frac{x^{\alpha-1}(1-x)^{b-1}}{Beta(\alpha,b)} dx \\ &= uv \; {}_2F_1\left(1, \alpha; \alpha+b; -(1-u)(1-v)\right) \end{aligned} \tag{3.21}$$

where parameter $\alpha > 0$, with limit cases $\lim_{\alpha\to 0} C'^{-}_{\alpha}(u,v;b) = \Pi$ and $\lim_{\alpha\to\infty} C'^{-}_{\alpha}(u,v;b) = C_{-1}(u,v)$.

Alternatively, a negative-coverage mixture copula can also be constructed by shifting the mixing Beta distribution as follows:

$$\begin{aligned} C_\Theta(u,v) \underset{\Theta}{\wedge} (X-1) &= \int_0^1 \frac{uv}{1-(x-1)(1-u)(1-v)} \frac{x^{\alpha-1}(1-x)^{b-1}}{Beta(\alpha,b)} dx \\ &= \frac{uv}{1+(1-u)(1-v)} \; {}_2F_1\left(1, \alpha; \alpha+b; \frac{(1-u)(1-v)}{1+(1-u)(1-v)}\right) \end{aligned}$$

where the Product copula Π corresponds to the limit case $\alpha \to \infty$, and $C_{-1}(u, v)$ to the limit case $\alpha \to 0$.

In addition to conveying prior information, informative mixing can also serve to vary the dependence structures covered by the mixture copula. As has been established above, the parent family and the mixture family do not encompass the same dependence structures. In particular, the AMH copula nests the Product copula as a special form, while the *Beta*-mixture family does not. Even at the point where the mixed copula generates zero dependence, that point does not represent independence; it is not Π. By contrast, both of the informatively mixed copulas (3.19) and (3.20) retain Π as a special case.

Hence, in addition to conveying prior information, informative mixing can also have the effect of "restoring" a particular dependence structure "lost" through ordinary mixing.